PROBABLY INTERSECTING FAMILIES

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A family $\mathcal{A}$ of sets is said to be *intersecting* if $A \cap B \neq \emptyset$ for all A, $B \in \mathcal{A}$. It is a well-known and simple fact that an intersecting family of subsets of $[n] = \{1, 2, \ldots, n\}$ can contain at most 2^{n-1} sets.

At RSA 2005, Katona, Katona and Katona asked the following question. Suppose instead $\mathcal{A} \subset \mathcal{P}[n]$ satisfies $|\mathcal{A}| = 2^{n-1} + i$ for some fixed $i > 0$. Create a new family $\mathcal{A}_p$ by choosing each member of $\mathcal{A}$ independently with some fixed probability p. How do we choose $\mathcal{A}$ to maximize the probability that $\mathcal{A}_p$ is intersecting?

They conjectured that there is a nested sequence of optimal families for $i = 1, 2, \ldots, 2^{n-1}$. We show that the families $[n]^{(\geqslant r)} = \{A \subset [n] : |A| \geqslant r\}$ are optimal for the appropriate values of i, thereby proving the conjecture for this sequence of values. Moreover, we show that for the intermediate values of i there exist optimal families lying between those that we have found. However, in further work joint with Mark Walters (Queen Mary, University of London), we show that the full conjecture is false for every value of p provided that n is sufficiently large: the optimal families of sizes between those of the families $[n]^{(\geqslant r)}$ are not always nested.

It turns out that the optimal families we find simultaneously maximize the number of intersecting subfamilies of every possible order.

Standard compression techniques appear inadequate to solve the problem as they do not preserve intersection properties of subfamilies. Instead, our main tool is a novel compression method, together with a way of 'compressing' subfamilies, which may be of independent interest.