The Size of the Largest Part in Weighted Partitions of Integers into Powers of Primes

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For a given sequence of non-negative numbers $b = \{b_k\}_{k\geq 1}$, we consider the partition function $p_b(n)$ defined by $1 + \sum_{n=1}^{\infty} p_b(n)x^n = \prod_{k=1}^{\infty}(1 - x^k)^{-b_k}$. It represents a count of the number of partitions of n into positive integer summands k weighted by the parameters b_k. A fairly general scheme of assumptions on the sequence b was proposed by G. Meinardus, Math. Z. 59(1954), 388-398, who established an asymptotic formula for $p_b(n)$ as $n \to \infty$. His approach is based on analytical properties of the Dirichlet generating series $D_b(s) = \sum_{k=1}^{\infty} b_k k^{-s}, s = \sigma + iy$. One of Meinardus conditions requires that $D_b(s)$ converges in the half-plane $\sigma > \rho > 0$ and there is a constant $C_0 > 0$, such that the function $D_b(s)$ has an analytical continuation to the half-plane $\sigma \geq -C_0$ on which it is analytic except for the simple pole at $s = \rho$ with a positive residue. This condition is satisfied by many important types of integer partitions. N. A. Brigham, Proc. Amer. Math. Soc. 1(1950), 192-204, proposed and studied a model of partitions with weights $b_k = \Lambda(k)$, where $\Lambda(p^r) = \log p$, p prime, and $\Lambda(k) = 0$ for all other values of k ($\Lambda(k)$ is also called von Mangoldt function). The Dirichlet generating series for these weights is $D_\Lambda(s) = -\zeta'(s)/\zeta(s)$, where ζ denotes the Riemann zeta function. Since the non-trivial zeta zeros are poles of $D_\Lambda(s)$, it does not satisfy Meinardus conditions. Let $p_\Lambda(n)$ be the corresponding weighted count of partitions of n into prime powers. Its asymptotic was determined by B. Richmond, Canad. J. Math. 27(1978), 1083-1091, and was subsequently improved by Y. Yang, Trans. Amer. Math. Soc. 352(2000), 2581-2600. Assuming that a weighted partition of n is selected with probability $1/p_\Lambda(n)$, we study the limiting distribution of the largest part size X_n as $n \to \infty$. As in the Meinardus case (see L. Mutafchiev, Combinatorics Probab. Comput. 22(2013), 433-454), we show that X_n, appropriately normalized, is approximately Gumbel distributed.